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A MATHEMATICAL THEORY OF GUARANTEE POLICIES

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A MATHEMATICAL THEORY OF GUARANTEE POLICIES

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DOCTOR OF PHILOSOPHY

By

Lloyd F. Bell

January 1961

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Donald J. Ferdman

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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A MATHEMATICAL THEORY OF GUARANTEE POLICIES

by

Lloyd F. Bell

CHAPTER I

INTRODUCTION AND SUMMARY

1. <u>Purpose</u> <u>and</u> <u>Scope</u>.

A guarantee is a contractual obligation incurred by a producer or vendor which is made in connection with the sale of an item or service. The guarantee stipulates that the producer or vendor agrees to remedy specified defects or failures of the commodity sold. The purpose of the guarantee is to promote sales by giving the prospective customer confidence that he will obtain satisfactory service from his purchase. There are many different varieties of guarantees including the "double your money back if you are not satisfied" offer, the frequently encountered scheme in which the purchase price is prorated over the time period of the guarantee, the offer to replace any initially defective parts with the customer being charged for labor and handling, and the agreement to replace the item without charge if any failure occurs within a specified period of time.

There is no technical literature available describing the procedures by which producers decide on the structure of the guarantees which they offer. The planning of a guarantee policy must involve the consideration of many factors both psychological and quantitative in nature. It seems likely that management generally feels that the analysis of guarantees should be made subjectively on the basis of experience rather

than by utilizing the more objective mathematical techniques of management science. Such an attitude on the part of management is described by Mayer [1] in his paper on replacement theory. Even though many mathematical results have been obtained in replacement theory, they are not widely used. Mayer suggests that such theoretical results deserve careful consideration by industry because they at least identify and evaluate the factors which are most significant in determining optimal procedures. It is believed that the same remark would apply to the theoretical study of guarantee policies. The purpose of this study is to investigate the mathematical structure of a class of guarantee policies and to derive criteria for the selection of an optimal guarantee policy within this class.

This study considers guarantee policies for items which fail under use or deteriorate with time, and for which the length of the guarantee is significantly related to the anticipated life of the item. Guarantees which run for a relatively short period of time and which are intended only to insure against an initially defective item are not considered. The sales volume or demand for such items will be affected by many factors such as price, quality, producer reputation and the availability of alternative products. It is assumed that where these effects are constant, the demand will be determined only by the nature of the associated guarantee policy. Although the maximization of profit may not always be the objective of a sales operation, it appears to be the most frequent one and can readily be characterized mathematically. Therefore, for the

purposes of this study, the maximization of profit is assumed to be the criterion by which management selects the guarantee policy for a product.

2. Guarantee Policies.

One commonly used guarantee policy operates in the following manner: if an item which is guaranteed for a time period of length x fails at time ξ after the beginning of the period, where $\xi < x$, the item is replaced, the guarantee is renewed, and the customer is charged the fraction ξ/x of the price of the item. If x is large compared to the mean failure time, most items will fail prior to the expiration of the guarantee. In this situation the operation of the guarantee is approximately equivalent to a leasing arrangement where the customer pays a constant rental rate for the use of the item.

A second commonly used policy differs from the previously described policy in that the customer is not charged when an item is replaced under the guarantee. If an item is guaranteed for a period of time x and fails at time $\xi < x$, it is replaced and the guarantee is renewed without any charge. Under this policy it is clear that x must be selected so that most of the items will not fail until after the expiration of the guarantee; otherwise, the cost of the replacement items will result in an overall loss to the producer.

The choice between these two policies and the selection of the length of guarantee period will depend on the production cost, sales price, distribution of the times of failure and the effects of these choices on demand. Consideration of this policy selection problem

immediately suggests a generalization which includes the previous two policies as special cases. This generalization may be described as follows: if an item is guaranteed for a period of time x and fails at time $\xi < x$, it is replaced and the guarantee is renewed with the customer being charged the fraction $k\xi/x$, $0 \leq k \leq 1$, of the price of the item. Thus $k = 1$ corresponds to the first policy described above and $k = 0$ to the second. In addition to the problem of selecting between the $k = 1$ and $k = 0$ policies, an interesting question arises as to whether there exist situations in which a policy corresponding to a value of k between 0 and 1 is superior to both the $k = 0$ and $k = 1$ policies.

Other formulations of generalized guarantee policies are of course possible. As an example, one alternative formulation includes as special cases the same $k = 1$ policy used above and a $k = 0$ policy which differs from the one above in that, when a replacement item is issued, the guarantee is not extended beyond the initial expiration time. This alternative general formulation is as follows: if an item is guaranteed for a period of time x and fails at time $\xi < x$, it is replaced and the guarantee is extended for a period $k\xi$, $0 \leq k \leq 1$, with the customer being charged the fraction $k\xi/x$ of the price of the item.

3. Summary.

In Chapters II through V, the first generalized guarantee policy defined is analyzed from the standpoint of earning maximum average rate of profit for the producer; in Chapter VI, the alternative policy is

considered with criteria for comparison of the associated $k = 0$ and $k = 1$ policies being derived. More specifically, Chapter II includes the mathematical derivation of the expected rate of profit as a function of the type and length of the guarantee, the demand, the production cost and the sales price. It also includes a discussion of the existence of an optimal guarantee policy and its possible uniqueness. In Chapter III, conditions are derived for the cases under which the optimal action is to offer no guarantee. In Chapter IV, conditions are derived for identifying the optimal type and length of guarantee for those cases in which a guarantee should be offered. Chapter V includes a step by step procedure for solving guarantee problems belonging to a class in which the failure distribution is continuous and the demand function is of the form $\varphi[g(k)x]$, where φ is a continuous non-decreasing function and g is a continuous non-increasing function. Also included is an example demonstrating the application of this procedure to the negative exponential failure distribution. The demand function is varied to illustrate conditions under which different specific guarantee policies are optimal.

CHAPTER II

MATHEMATICAL STRUCTURE

1. Expected Rate of Profit Per Customer.

If a single customer uses a product sold under the previously described guarantee policy for a long period of time and he replaces each item immediately upon failure, a renewal sequence is generated. The expected rate of profit for the operation follows directly from well-known results in renewal theory cited by Smith [2].

Consider a sequence of items issued at times $t_1, t_2, \dots, t_i, \dots$ with corresponding lifetimes of lengths $\xi_1, \xi_2, \dots, \xi_i, \dots$. Then, $\xi_1 = t_2 - t_1$, $\xi_2 = t_3 - t_2, \dots, \xi_i = t_{i+1} - t_i, \dots$, as shown in Figure 1.

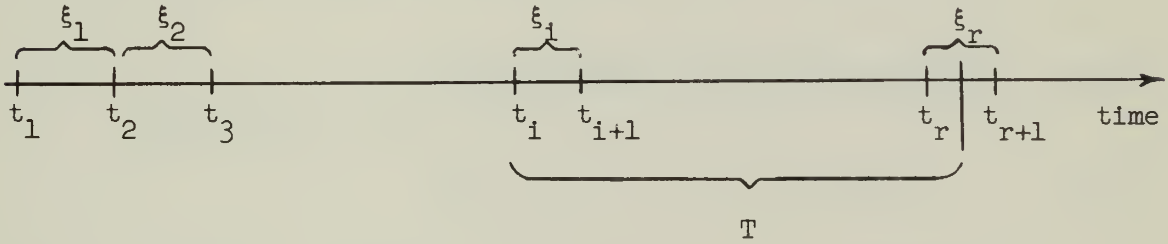

Figure 1

The $\{\xi_j\}$ are independent random variables which are identically distributed with common distribution function $F(\xi)$, where $F(0) = 0$. At the time of each sale or issue, a profit is obtained which is equal to the amount received less the production cost of the item. At t_2

and subsequent times, the amount received depends on the life of the previous item. Let Y_j be the profit obtained at time t_j. The $\{Y_j\}$ are independent, identically distributed random variables; Y_j and ξ_h are independent for all $h \neq j-1$; and Y_j and ξ_{j-1} have a joint probability distribution with non-zero covariance.

If $H_i(T)$ is the sum of the profits obtained during a time period of length T commencing at t_i, and $N \geq i$ is determined by $t_N < t_i + T \leq t_{N+1}$,

$$H_i(T) = \begin{cases} 0 \,, & N = i \text{ or } \xi_i \geq T \\ Y_{i+1} + Y_{i+2} + \cdots + Y_N \,, & N > i \text{ or } \xi_i < T \,. \end{cases}$$

For $\xi_i < T$, $Y_{i+2} + Y_{i+3} + \cdots + Y_N = H_{i+1}(T-\xi_i)$; therefore,

$$E\{H_i(T) | \xi_i = u\} = \begin{cases} E\{Y_{i+1} | \xi_i = u\} + E\{H_{i+1}(T-\xi_i) | \xi_i = u\}, & u < T \\ 0 \,, & u \geq T. \end{cases}$$

Because $H_{i+1}(T-\xi_i)$ depends on ξ_i only through the length of the interval $T-\xi_i$, $E\{H_{i+1}(T-\xi_i) | \xi_i = u\} = E\{H_{i+1}(T-u)\}$ for $u < T$. Then,

$$E\{H_i(T)\} = \int_0^\infty E\{H_i(T) | \xi_i = u\} \, dF(u)$$

$$= \int_0^T E\{Y_{i+1} | \xi_i = u\} \, dF(u) + \int_0^T E\{H_{i+1}(T-u)\} \, dF(u) \,.$$

If $V_i(T)$ is defined as

$$V_i(T) = \int_0^T E\{Y_{i+1} | \xi_i = u\} \, dF(u) \ ,$$

$V_i(T)$ is a non-negative, increasing function of T and $\lim_{T \to \infty} V_i(T)$ is finite. The $\{\xi_i\}$ and $\{Y_i\}$ are both sequences of independent, identically distributed random variables; therefore, $E\{H_{i+1}(T-u\} = E\{H_i(T-u)\}$. Since the quantities $V_i(T)$ and $E\{H_i(T-u)\}$ do not depend on i, the subscripts can be dropped and

$$E\{H(T)\} = V(T) + \int_0^T E\{H(T-u)\} \, dF(u) \ . \tag{1.1}$$

This is a generalized renewal equation as discussed by Karlin [3]. If $M(T)$ is the renewal function defined by

$$M(T) = \sum_{n=1}^{\infty} F^{(n)}(T) \ ,$$

where $F^{(n)}$ is the n-fold convolution of F, then (1.1) has the solution

$$E\{H(T)\} = V(T) \cdot [1 + M(T)] \ .$$

The expected rate of profit during the period T for a single customer is $E\{H(T)\}/T$, and the expected long term average rate of profit per single customer is given by

$$R_1 = \lim_{T \to \infty} \frac{E\{H(T)\}}{T}$$

$$= \lim_{T \to \infty} V(T) \frac{1 + M(T)}{T} \quad .$$

Feller [4] shows that $\lim_{T \to \infty} \frac{M(T)}{T} = \frac{1}{E\{\xi\}}$; therefore,

$$R_1 = \frac{1}{E(\xi)} \lim_{T \to \infty} V(T) \quad .$$

For the generalized guarantee policy being considered, the profits obtained will be

$$Y_1 = \quad C - C_o$$

$$Y_j = \begin{cases} \frac{kC}{x} \xi_{j-1} - C_o, & \xi_{j-1} < x \\ \\ C - C_o, & \xi_{j-1} \geq x \end{cases}$$

for $j = 2, 3, \ldots ,$ where

C = sales price of each item

C_o = production cost of each item

x = length of the guarantee

k = guarantee parameter.

Since Y_{i+1} is a deterministic function of ξ_i,

$$E\{Y_{i+1} | \xi_i = u\} = \begin{cases} \frac{kC}{x} u - C_o \ , & u < x \\ C - C_o \ , & u \geq x \ , \end{cases}$$

$$V(T) = V_i(T) = \int_0^T E\{Y_{i+1} | \xi_i = u\} \, dF(u), \quad \text{for } i = 1, 2, \ldots$$

$$= \int_0^x \left(\frac{kC}{x} u - C_o\right) dF(u) + \int_x^T (C - C_o) \, dF(u)$$

$$= \frac{k}{x} \int_0^x u dF(u) + (C - C_o) \, F(T) - CF(x) \ ,$$

and

$$\lim_{T \to \infty} V(T) = \frac{kC}{x} \int_0^x u dF(u) + C[1 - F(x)] - C_o \ . \tag{1.2}$$

If C_o is expressed as a fraction of C, the C can be factored from each term of (1.2) and, without loss of generality, can be assumed to be unity. All expected profits in the results obtained are then to be scaled according to the price per item. Similarly, if x is expressed as a multiple of the mean of the failure distribution, $E\{\xi\}$, this mean can also be assumed to be unity. Finally, let

$$G(x) = \int_0^x u dF(u) \ . \tag{1.3}$$

Then the expected rate of profit per customer, R_1, can be written as a function of k and x as follows:

$$R_1(k,x) = 1 - C_o + \frac{k}{x} G(x) - F(x) \; . \tag{1.4}$$

2. Expected Rate of Overall Profit.

The expected rate of overall profit for the sale of a product, $R(k,x)$, will be the profit per customer times the demand. Under the assumptions given in Chapter I, the demand is determined by the type and length of the associated guarantee and will be designated $D(k,x)$. Properties of $D(k,x)$ will be discussed in the next section. Then

$$R(k,x) = R_1(k,x) \cdot D(k,x) \; . \tag{2.1}$$

The problem of choosing a guarantee policy will be that of finding the particular values of k and x, say K and x_K, which maximize $R(k,x)$.

From equation (1.4), it is seen that for any k, $0 \leq k \leq 1$, $R_1(k,0) = 1 - C_o > 0$ and $R_1(k,\infty) = -C_o$. For $x' < x''$,

$$\begin{aligned} R_1(k,x'') - R_1(k,x') &= \left\{ \frac{k}{x''} \int_0^{x''} u\,dF(u) - F(x'') \right\} - \left\{ \frac{k}{x'} \int_0^{x'} u\,dF(u) - F(x') \right\} \\ &= \int_0^{x''} \left(\frac{ku}{x''} - 1\right) dF(u) - \int_0^{x'} \left(\frac{ku}{x'} - 1\right) dF(u) \\ &< \int_0^{x''} \left(\frac{ku}{x''} - 1\right) dF(u) - \int_0^{x'} \left(\frac{ku}{x''} - 1\right) dF(u) \\ &= \int_{x'}^{x''} \left(\frac{ku}{x''} - 1\right) dF(u) \leq 0 \; , \qquad \text{for } k \leq 1 \; . \end{aligned}$$

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Therefore, $R_1(k,x)$ is a non-increasing function of x, and it follows that for any k, $0 \leq k \leq 1$, there exists an x_k^o such that $R_1(k,x) > 0$, for $0 \leq x < x_k^o$, and $R_1(k,x) \leq 0$ for $x_k^o \leq x$. Then, if $D(k,x)$ is non-negative, $R(k,x) \leq 0$, for $x_k^o \leq x$, and the search for (K,x_K) can be restricted to the set $S_1 = \{(k,x): 0 \leq k \leq 1, 0 \leq x \leq x_k^o\}$; because, outside S_1, no profit is to be expected.

3. Demand Functions.

Any demand function, $D(k,x)$, will be assumed to satisfy the following conditions:

(a) $D(k,x)$ is continuous and non-decreasing in x;

(b) $D(k,x)$ is continuous and non-increasing in k;

(c) $D(k,0)$ is non-negative and independent of k.

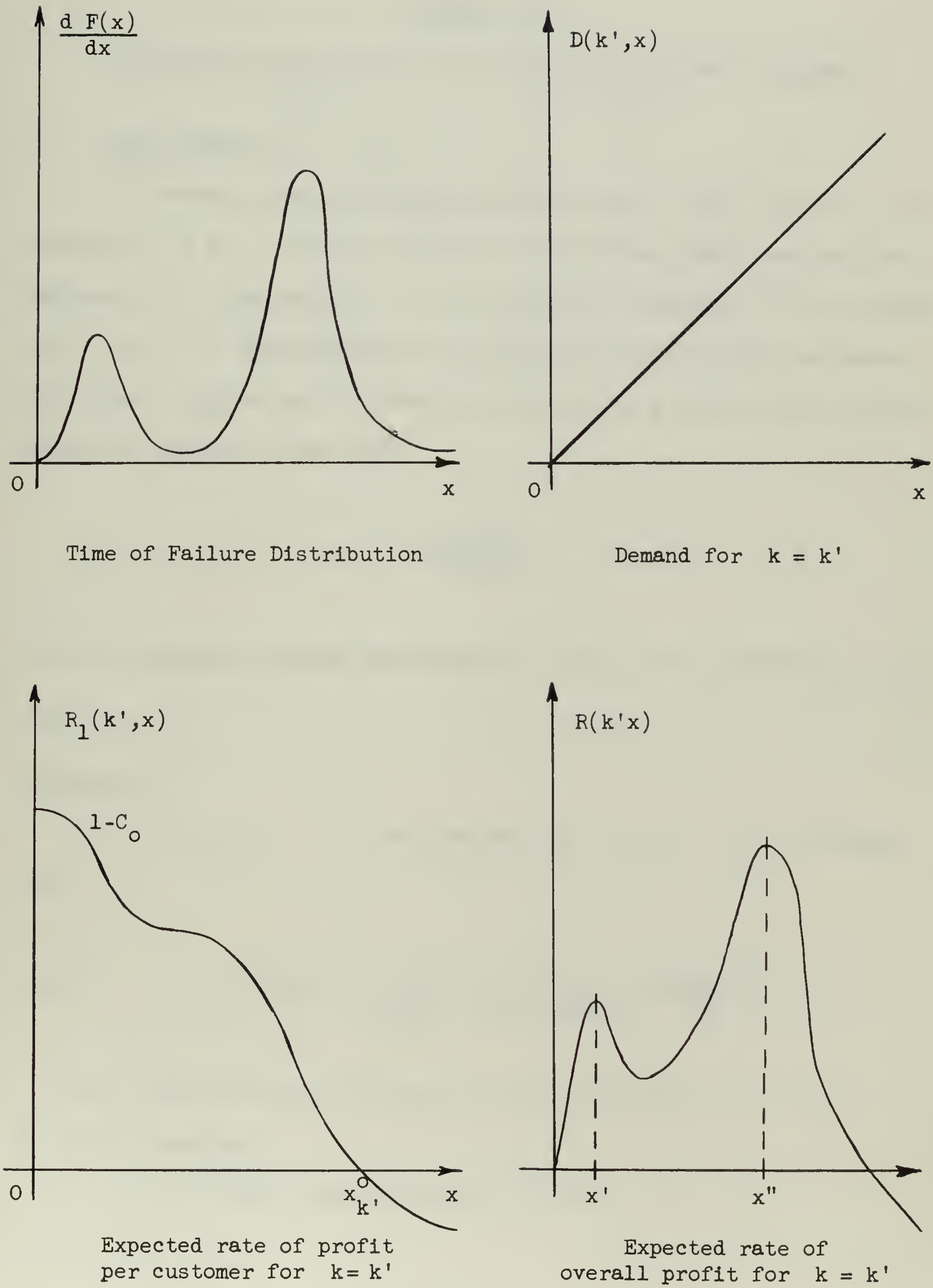
It is noted that (a) and (c) imply that $D(k,x) \geq 0$ for all (k,x). These conditions contain no restrictions which would rule out any demand function which would be encountered in an actual sales operation.

Let $S_2 = \{(k,x): D(k,x) > 0\}$. Then no profit can be expected outside S_2 and the intersection $S = S_1 \cap S_2$. Clearly, if S is empty and no profit is expected, the product should not be offered for sale. While it is unlikely that the solution of a reasonable guarantee problem will reveal such a situation, rejection of these cases permits consideration of only those cases where a positive maximum exists for $R(k,x)$. In the ensuing sections, S is assumed to be non-empty, and all (k,x) referred to are defined to be in $\overline{S}$, the closure of S.

$R(k,x)$ will be bounded on $\bar{S}$ by $(1-C_0) \max_S D(k,x)$. If $F(\xi)$ is a continuous distribution, $R_1(k,x)$ and $R(k,x)$ will be continuous. $R(k,x)$ will then attain its maximum on $\bar{S}$ and therefore on S, because $R(k,x) \leq 0$ for (k,x) in $\bar{S} - S$. Also, if for any fixed k', $0 \leq k' \leq 1$, $S_{k'} = \{x: (k',x) \in S\}$ and $F(\xi)$ is continuous, $R(k',x)$ will attain a maximum on $S_{k'}$. Let $x_{k'}$ be defined such that $R(k',x_{k'}) = \max_{S_{k'}} R(k',x)$. The $x_{k'}$ is not necessarily unique.

4. Multiplicity and Uniqueness of Local Maxima.

If $R(k,x)$ has only one local maximum, its location, (K,x_K), can be determined by the usual analytic methods as given by Courant [5]. In general, several local maxima may exist; and, if so, each must be located and evaluated in order to identify the location of the absolute maximum. An example of the complex case is shown in Figure 2. The values of the maxima $R(k,x')$ and $R(k,x")$ must be evaluated in order to determine which is greater. Similar maxima also occur for other values of k, so that the surface $R(k,x)$ will contain two or more local maxima for this case. Criteria for local maxima occurring at $x = 0$, $k = 0$, $k = 1$ and intermediate k will be developed in the next chapters. Sufficient conditions for these local maxima to be unique will also be given.



Time of Failure Distribution

Demand for $k = k'$

Expected rate of profit per customer for $k = k'$

Expected rate of overall profit for $k = k'$

Figure 2

CHAPTER III

CONDITIONS UNDER WHICH IT IS OPTIMAL NOT TO GIVE A GUARANTEE

1. Local Maximum at $x = 0$.

The necessary and sufficient conditions that $R(k,x)$ have a local maximum at $x = 0$ is that its first non-vanishing right-hand partial derivative with respect to x at that point be negative. It is assumed that $R(k,x)$ is differentiable as necessary to perform the indicated operations. Right- and left-hand derivatives at a point will be represented by symbols of the form

$$\frac{\partial R(k,a^+)}{\partial x}$$

for the right-hand partial derivative of $R(k,x)$ with respect to x at $x = a$.

Theorem 1.1.

(a) If $R(\cdot,0)$ is a local maximum of $R(\cdot,x)$, it is necessary that

$$\frac{1}{1-C_o} \frac{dF(0^+)}{dx} \geq \max_{0 \leq k \leq 1} \frac{2}{(2-k)\,D(k,0)} \frac{\partial D(k,0^+)}{\partial x} . \tag{1.1}$$

(b) The following conditions are each sufficient for $R(\cdot,0)$ to be a local maximum:

(1) Strict inequality holds in (1.1);

(2) (1.1) holds, $\frac{dF(0^+)}{dx} > 0$, $k > 0$ and

$$\frac{\partial D(0,0^+)}{\partial x} < \max_{0<k\leq 1} \frac{2}{2-k} \frac{\partial D(k,0^+)}{\partial x} ;$$

(3) (1.1) holds, $\frac{dF(0^+)}{dx} = 0$ or $k = 0$, and

$$\frac{1}{1-C_o} \frac{d^2F(0^+)}{dx^2} > \frac{3}{(3-2k')\, D(0,k')} \frac{\partial^2 D(k',0^+)}{\partial x^2} \tag{1.2}$$

for all k' such that

$$\frac{2}{2-k'} \frac{\partial D(k',0^+)}{\partial x} = \max_{0\leq k\leq 1} \frac{2}{2-k} \frac{\partial D(k,0^+)}{\partial x} .$$

Proof.

(a) It is necessary that $\frac{\partial R(\cdot,0^+)}{\partial x} \leq 0$ if $R(\cdot,0)$ is a local maximum, and strict inequality is sufficient because $R(k,0) = (1-C_o)\, D(k,0)$ is independent of k. Now

$$0 \geq \frac{\partial R(k,0^+)}{\partial x} = R_1(k,0) \frac{\partial D(k,0^+)}{\partial x} + \frac{\partial R_1(k,0^+)}{\partial x} D(k,0)$$

$$= (1-C_o) \frac{\partial D(k,0^+)}{\partial x} + [-(1 - \frac{k}{2}) \frac{dF(0^+)}{dx}]\, D(k,0)$$

or, rearranging,

$$\frac{1}{1-C_o} \frac{dF(0^+)}{dx} \geq \frac{2}{(2-k)\, D(k,0)} \frac{\partial D(k,0^+)}{\partial x} , \quad \text{for } 0 \leq k \leq 1 .$$

Taking the maximum over k of the right side of this inequality gives the necessary condition and (1) of the sufficient conditions.

(b) Next it is necessary to consider second order derivatives for those values of k for which equality holds in (1.1). First,

$$\frac{\partial^2 R_1(k,0^+)}{\partial x^2} = \lim_{x \to 0^+} \left\{ \frac{2k}{k^3} \int_0^x \xi \, dF(\xi) - \frac{k}{x} \frac{dF(x)}{dx} - (1-k) \frac{d^2 F(x)}{dx^2} \right\}$$

$$= \lim_{x \to 0^+} \left\{ - \frac{k}{3x} \frac{dF(x)}{dx} - (1-k) \frac{d^2 F(x)}{dx^2} \right\}$$

$$= \begin{cases} -\infty \; ; \; k > 0 \, , \; \dfrac{dF(0^+)}{dx} > 0 \\ -(1 - \frac{2k}{3}) \dfrac{d^2F(0^+)}{dx^2} \; ; \; \dfrac{dF(0^+)}{dx} = 0 \quad \text{or} \quad k = 0 \, . \end{cases}$$

Then for $k > 0$ and $\frac{dF(0^+)}{dx} > 0$,

$$\frac{\partial^2 R(k,0^+)}{\partial x^2} = R_1(k,0) \frac{\partial^2 D(k,0^+)}{\partial x^2} + 2 \frac{\partial R_1(k,0^+)}{\partial x} \frac{\partial D(k,0^+)}{\partial x} + \frac{\partial^2 R_1(k,0^+)}{\partial x^2} D(k,0)$$

$$= - \infty \, ,$$

because $D(k,0) > 0$, $\frac{\partial^2 R_1(k,0^+)}{\partial x^2} = - \infty$, and the other terms are all finite. If $k = 0$ is not one of the values for which equality holds in (1.1), $\frac{dF(0^+)}{dx} > 0$ is then sufficient, proving (2).

(c) If $\frac{dF(0^+)}{dx} = 0$ or $k = 0$, then for those k for which equality holds in (1.1), it is sufficient if

$$0 > \frac{\partial^2 R(k,0^+)}{\partial x^2} = (1-C_o) \frac{\partial^2 D(k,0^+)}{\partial x^2} - (1 - \frac{2k}{3}) \frac{d^2 F(0^+)}{dx^2} D(k,0)$$

or

$$\frac{1}{1-C_o} \frac{d^2 F(0^+)}{dx^2} > \frac{3}{(3-2k)\ D(k,0)} \frac{\partial^2 D(k,0^+)}{\partial x^2},$$

thereby giving (3).

Corollary 1.1.1. If $\frac{dF(0^+)}{dx} = 0$ and $\frac{\partial D(k,0^+)}{\partial x} > 0$ for some k, then $x = 0$ is not optimal.

Proof. These conditions violate the necessary conditions given in (a) of the theorem. If $R(\cdot,0)$ is not a local maximum, $x = 0$ cannot be optimal.

If equality holds in expression (1.2) as well as (1.1) for some values of k', it is necessary to consider still higher order derivatives. This leads to

Theorem 1.2. If

$$D(k,0) > 0,\ \frac{\partial D(k,0^+)}{\partial x} = \cdots = \frac{\partial^{j-1} D(k,0^+)}{\partial x^{j-1}} = \frac{dF(0^+)}{dx} = \cdots = \frac{d^{j-1} F(0^+)}{dx^{j-1}} = 0,$$

then

$$(1.3) \qquad \frac{1}{1-C_o} \frac{d^j F(0^+)}{dx^j} \geq \max_{0 \leq k \leq 1} \frac{j+1}{(j+1-jk)\ D(k,0)} \frac{\partial^j D(k,0^+)}{\partial x^j}$$

is necessary for $R(\cdot,0)$ to be a local maximum and strict inequality is sufficient.

Proof. The vanishing derivatives result in $\frac{\partial^i R(k,0^+)}{\partial x^i} = 0$ for $i = 1, 2, \ldots, j-1$ and (1.3) is equivalent to $\frac{\partial^j R(k,0^+)}{\partial x^j} \leq 0$, $0 \leq k \leq 1$, which is the required necessary condition with strict inequality being sufficient.

2. $x = 0$ Optimal.

If $R(\cdot,0)$ is a local maximum of $R(\cdot,x)$, the additional necessary and sufficient condition for $x = 0$ to be optimal is simply that there not exist another local maximum of greater magnitude.

Theorem 2.1. If $R(\cdot,0)$ is a local maximum and $\frac{\partial R(k,x)}{\partial x} \leq 0$ for all (k,x) in S, then $x = 0$ is optimum and $R(\cdot,0)$ is the maximum.

Proof. If $R(\cdot,0)$ is a local maximum, the condition $\frac{\partial R(k,x)}{\partial x} \leq 0$ for all (k,x) in S is sufficient to insure that no other local maxima exist and the result follows.

Actually the conditions of this theorem are restrictive and will not be met by many cases for which $k = 0$ is actually optimal. All cases for which $R(\cdot,0)$ is the greater of two or more local maxima are not included.

CHAPTER IV

OPTIMALITY CONDITIONS FOR GUARANTEE POLICIES

1. Differentiability Conditions and Structure of the Demand Function.

For $R(k,x_k)$ to be a local maximum, it is necessary that $\frac{\partial R(k,x_k^-)}{\partial x} \geq 0$ and $\frac{\partial R(k,x_k^+)}{\partial x} \leq 0$, with appropriate conditions on higher order derivatives if equality holds. It can be assumed that the failure distribution does not have a discrete component. Furthermore, any distribution with a discontinuous density function can be approximated as closely as desired by a distribution with a continuous, differentiable density function. Therefore, in order to avoid the tedium of considering the right- and left-hand derivatives in the paragraphs to follow, it will be assumed that $F(\xi)$ has a continuous density function $f(y)$ and that $f(y)$ is differentiable as necessary to perform the indicated operations.

Consideration of condition 3(c) of Chapter II suggests that k and x act in a multiplicative manner in the demand function, giving the representation

$$D(k,x) = \varphi[g(k) \cdot h(x)] ,$$

where the following conditions hold: φ and h are continuous, non-negative, non-decreasing functions; $h(0) = 0$; and g is a continuous, positive, non-increasing function. Consideration will be restricted to the case where $h(x) = x$, and it will be assumed that φ and g are differentiable as necessary to perform the indicated operations. Then

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$$D(k,x) = \varphi[g(k)x] \tag{1.1}$$

$$\frac{\partial D(k,x)}{\partial x} = g(k)\ \varphi'[g(k)x] \tag{1.2}$$

$$\begin{aligned} \frac{\partial D(k,x)}{\partial k} &= x\ g'(k)\ \varphi'[g(k)x] \\ &= x\ \frac{g'(k)}{g(k)}\ \frac{\partial D(k,x)}{\partial x}\ . \end{aligned} \tag{1.3}$$

One necessary condition for $R(k,x_k)$ to be a local maximum of $R(k,x)$ is that $\dfrac{\partial R(k,x_k)}{\partial x} = 0$; this requires that x_k satisfy

$$\frac{\dfrac{k}{x_k^2}\ G(x_k) + (1-k)\ f(x_k)}{1-C_o + \dfrac{k}{x_k}\ G(x_k) - F(x_k)} = \frac{g(k)\ \varphi'[g(k)x_k]}{\varphi[g(k)x_k]}\ , \tag{1.4}$$

with the appropriate additional conditions on the higher order derivatives.

A similar condition holds in the k dimension, except that here the maximum can occur against a boundary, $k = 0$ or $k = 1$, in which case $\dfrac{\partial R(k,x_k)}{\partial k}$ may be strictly less than or greater than 0.

2. Optimality Conditions for $k = 0$.

For a local maximum to occur at the $k = 0$ boundary, the first non-vanishing derivative with respect to k must be negative at $(0,x_o)$.

<u>Theorem 2.1</u>. For $R(0,x_o)$, $x_o > 0$, to be a local maximum of $R(k,x)$, it is necessary that

$$\frac{G(x_o)}{x_o^2\, f(x_o)} \leq - \frac{g'(0)}{g(0)}\,, \tag{2.1}$$

and strict inequality is sufficient.

<u>Proof</u>. Since $R(0,x_o) = \max\limits_{0 \leq x} R(0,x)$, it follows that $\dfrac{\partial R(0,x_o)}{\partial k} \leq 0$ is necessary for $R(0,x_o)$ to be a local maximum, and strict inequality is sufficient.

$$0 \geq \frac{\partial R(0,x_o)}{\partial k} = R_1(0,x_o)\, \frac{\partial D(0,x_o)}{\partial k} + \frac{\partial R_1(0,x_o)}{\partial k}\, D(0,x_o)\ ;$$

and on substituting for $\dfrac{\partial D(0,x_o)}{\partial k}$ from (1.3),

$$\frac{\partial R(0,x_o)}{\partial k} = \frac{x_o\, g'(0)}{g(0)}\, R_1(0,x_o)\, \frac{\partial D(0,x_o)}{\partial x} + \frac{\partial R_1(0,x_o)}{\partial k}\, D(0,x_o)\ .$$

Since $\dfrac{\partial R(0,x_o)}{\partial x} = 0$, it follows that

$$\frac{\partial D(0,x_o)}{\partial x} = - \frac{\partial R_1(0,x_o)}{\partial x}\, \frac{D(0,x_o)}{R_1(0,x_o)}\ .$$

Therefore,

$$0 \geq \frac{x_o\, g'(0)}{g(0)}\, R_1(0,x_o) \left[- \frac{\partial R_1(0,x_o)}{\partial x}\, \frac{D(0,x_o)}{R_1(0,x_o)} \right] + \frac{\partial R_1(0,x_o)}{\partial k}\, D(0,x_o)$$

$$\geq D(0,x_o) \left\{ \frac{\partial R_1(0,x_o)}{\partial k} - \frac{x_o\, g'(0)}{g(0)}\, \frac{\partial R_1(0,x_o)}{\partial x} \right\}\,,$$

and

$$-\frac{\frac{\partial R_1(0,x_o)}{\partial k}}{x_o \frac{\partial R_1(0,x_o)}{\partial x}} < -\frac{g'(0)}{g(0)} .$$

Differentiating (1.4) of Chapter II gives

$$\frac{\partial R_1(k,x)}{\partial k} = \frac{1}{x} G(x) \tag{2.2}$$

and, substituting from (2.2) of Chapter II, the result follows.

If equality holds in (2.1), a local maximum with horizontal tangent plane may occur at $(0,x_o)$, so that it will be necessary to consider the second order partial derivative with respect to k. It is noted that the conditions of Theorem 2.1 depend on C_o and the form of φ only through the location of x_o. To find x_o, it will be necessary to solve (1.4) which will generally be a transcendental equation requiring numerical methods. This suggests the following somewhat more restrictive but much more easily applied result.

Corollary 2.1.1. If, for all $x \geq 0$,

$$\frac{G(x)}{x^2 f(x)} < -\frac{g'(0)}{g(0)} , \tag{2.3}$$

it follows that $R(0,x_o)$ is a local maximum of $R(k,x)$.

Proof. If the condition holds for all $x \geq 0$, it holds for x_o, and the theorem is satisfied.

<u>Corollary 2.1.2</u>. If, for $x_o > 0$,

$$(2.4) \qquad \frac{G(x_o)}{x_o^2 f(x_o)} > - \frac{g'(0)}{g(0)} ,$$

it follows that $k = 0$ cannot be optimal.

<u>Proof</u>. The given condition is equivalent to $\frac{\partial R(0,x_o)}{\partial k} > 0$; therefore, no local maximum can exist for $k = 0$, and $k = 0$ cannot be optimal. The following result is immediate.

<u>Corollary 2.1.3</u>. If, for all $x > 0$,

$$(2.5) \qquad \frac{G(x)}{x^2 f(x)} > - \frac{g'(0)}{g(0)} ,$$

then $k = 0$ cannot be optimal.

If $R(0,x_o)$ is a local maximum of $R(k,x)$, a sufficient condition that no greater maximum occur is that $\frac{\partial R(k,x_k)}{\partial k} \leq 0$, for all $k > 0$. This condition means that no other local maxima occur and will not include the cases in which $R(0,x_o)$ is the greater of two or more local maxima.

<u>Theorem 2.2</u>. If $R(0,x_o)$ is a local maximum of $R(k,x)$ and, for $0 < k \leq 1$,

$$(2.6) \qquad \frac{G(x_k)}{k\, G(x_k) + (1-k)x_k^2\, f(x_k)} \leq - \frac{g'(k)}{g(k)} ,$$

then $k = 0$ is optimal.

Proof. Proceeding exactly as in the proof of Theorem 2.1, the given conditions are seen to be equivalent to $\frac{\partial R(k,x_k)}{\partial k} \leq 0$, which is sufficient.

Corollary 2.2.1. If $R(0,x_o)$ is a local maximum and, for all (k,x) in S,

$$\frac{G(x)}{k\,G(x) + (1-k)x^2\,f(x)} \leq -\frac{g'(k)}{g(k)}\,, \tag{2.7}$$

then $k = 0$ is optimal.

Proof. If the condition holds for all (k,x) in S, it holds for (k,x_k) and the hypotheses of the theorem are satisfied.

Even though Theorem 2.2 or its corollary does not hold for all k in the interval $(0,1]$, it may hold over some interval $(0,k_o)$. If so, $R(0,x_o)$ is larger than $R(k,x)$ for k in the interval $(0,k_o)$ and no local maxima can occur there.

3. Optimality Conditions for $k = 1$.

The necessary and sufficient conditions for $R(1,x_1)$ to be a local maximum of $R(k,x)$ and $k = 1$ to be optimal are the same as the corresponding conditions for $k = 0$ with the inequalities being in the opposite sense. Proofs of the theorems are exactly the same; therefore the full details are not given.

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<u>Theorem 3.1</u>. For $R(1,x_1)$, $x_1 > 0$, to be a local maximum of $R(k,x)$, it is necessary that

$$(3.1) \qquad g(1) + g'(1) \geq 0 ,$$

and strict inequality is sufficient.

<u>Proof</u>. The given condition is equivalent to $\frac{\partial R(1,x_1)}{\partial k} \geq 0$. On substituting for $\frac{\partial D(1,x_1)}{\partial k}$ from (1.3) and for $\frac{\partial D(1,x_1)}{\partial x}$ from the equation $\frac{\partial R(1,x_1)}{\partial x} = 0$, it follows that

$$0 \leq \frac{\partial R(1,x_1)}{\partial k} = D(1,x_1)\left\{\frac{\partial R_1(1,x_1)}{\partial k} - \frac{x_1 g'(1)}{g(1)} \; \frac{\partial R_1(1,x_1)}{\partial x}\right\} .$$

Therefore,

$$- \frac{\dfrac{\partial R_1(1,x_1)}{\partial k}}{x_1 \dfrac{\partial R_1(1,x_1)}{\partial x}} \geq \frac{g'(1)}{g(1)} ;$$

and, since

$$\frac{\partial R_1(1,x_1)}{\partial k} = \frac{1}{x_1} G(x_1) = x_1 \; \frac{\partial R_1(1,x_1)}{\partial x} ,$$

the result follows.

Corollary 3.1.1. If

$$g(1) + g'(1) < 0 \ , \tag{3.2}$$

$k = 1$ cannot be optimal.

Proof. This condition is equivalent to $\dfrac{\partial R(1,x_1)}{\partial k} < 0$; therefore, no local maximum of $R(k,x)$ can occur for $k = 1$, and $k = 1$ cannot be optimal.

Theorem 3.2. If $R(1,x_1)$ is a local maximum of $R(k,x)$, and, for $k < 1$,

$$\frac{G(x_k)}{k\ G(x_k) + (1-k)x_k^2\ f(x_k)} \geq -\ \frac{g'(k)}{g(k)} \ , \tag{3.3}$$

then $k = 1$ is optimal.

Proof. The given condition is equivalent to $\dfrac{\partial R(k,x_k)}{\partial k} \geq 0$, for $k < 1$, which means that no local maxima other than $R(1,x_1)$ can occur.

Corollary 3.2.1. If $R(1,x_1)$ is a local maximum and, for all (k,x) in S,

$$\frac{G(x)}{k\ G(x) + (1-k)x^2\ f(x)} \geq -\ \frac{g'(k)}{g(k)} \ , \tag{3.4}$$

then $k = 1$ is optimal.

Proof. The given condition includes the condition of the theorem.

[illegible]

Corollary 3.2.2. If $\int_0^x y^2 f'(y)\, dy \leq 0$ for $x > 0$ and

$$(3.5) \qquad g(k) + (2-k)\, g'(k) > 0 , \qquad 0 \leq k \leq 1 ,$$

then $k = 1$ is optimal.

Proof. From the previous corollary,

$$\frac{G(x)}{k\, G(x) + (1-k)x^2 f(x)} = \frac{1}{k + (1-k)\,\dfrac{x^2 f(x)}{G(x)}}$$

$$= \frac{1}{k + (1-k)\,\dfrac{2x^2 f(x)}{x^2 f(x) - \int_0^x y^2 f'(y)dy}} ,$$

since

$$G(x) = \int_0^x y\, f(y)\, dy = \frac{1}{2}\, [x^2 f(x) - \int_0^x y^2 f'(y)\, dy] .$$

Then, if $\int_0^x y^2 f'(y)\, dy \leq 0$,

$$\frac{G(x)}{k\, G(x) + (1-k)\, x^2 f(x)} \geq \frac{1}{2-k} ,$$

so that this condition includes that of the previous corollary.

Corollary 3.2.2 is significant in that the integral condition is satisfied by a large class of failure distributions. Any distribution with a non-increasing density function will satisfy the condition and so

will any distribution which is initially decreasing and then does not increase too much.

If Theorem 3.2 or one of its corollaries holds for k in an interval $(k_1,1)$, but not in $(0,k_1)$, $k = 1$ may not be optimal. However, $R(1,x_1)$ is larger than $R(k,x)$ for k in the interval $(k_1,1)$, and no local maxima can occur there.

4. Conditions Under Which k, $0 < k < 1$, is Optimal.

The necessary and sufficient conditions for a local maximum for $0 < k < 1$ do not involve a boundary of the region S; therefore they usually require consideration of higher order derivatives than the $x = 0$, $k = 0$ and $k = 1$ cases. In simplest form, the conditions for $R(k,x)$ to have a local maximum at (K,x_K), $0 < K < 1$, are

$$(4.1) \qquad \frac{\partial R(K,x_K)}{\partial x} = \frac{\partial R(K,x_K)}{\partial k} = 0 ,$$

$$(4.2) \qquad \frac{\partial^2 R(K,x_K)}{\partial x^2} < 0 , \quad \frac{\partial^2 R(K,x_K)}{\partial k^2} < 0 ,$$

$$(4.3) \qquad \frac{\partial^2 R(K,x_K)}{\partial x^2} \cdot \frac{\partial^2 R(K,x_K)}{\partial k^2} - \left[\frac{\partial^2 R(K,x_K)}{\partial x\, \partial k} \right]^2 > 0 .$$

A further condition which is sufficient for such K to be optimal is that no local maxima occur for other k; this requires that

$$(4.4) \qquad \frac{\partial R(k,x_k)}{\partial k} \begin{cases} \geq 0, & 0 \leq k < K \\ \leq 0, & K < k \leq 1 . \end{cases}$$

The equations (4.1) are the same as (1.4) and (2.6) with equality holding; i.e.,

$$(4.5)\qquad \begin{cases} \dfrac{\dfrac{K}{x_K^2} G(x_K) + (1-K)\, f(x_K)}{1-C_o + \dfrac{K}{x_K} G(x_K) - F(x_K)} = \dfrac{g(K)\, \varphi'[g(K)x_K]}{\varphi[g(K)x_K]} \\[2em] \dfrac{G(x_K)}{K\, G(x_K) + (1-K)x_K^2\, f(x_K)} = -\dfrac{g'(K)}{g(K)} \quad . \end{cases}$$

Except in special cases, these simultaneous equations will be transcendental, and solution for K and x_K will require numerical methods. After K and x_K have been obtained, (4.2) and (4.3) can be confirmed by evaluating the following:

$$(4.6)\qquad \frac{\partial^2 R(K,x_K)}{\partial x^2} = \left\{1-C_o + \frac{K}{x_K} G(x_K) - F(x_K)\right\} g^2(K)\, \varphi''[g(K)x_K]$$
$$- 2\left\{\frac{K}{x_K^2} G(x_K) + (1-K)\, f(x_K)\right\} g(K)\, \varphi'[g(K)x_K]$$
$$+ \left\{\frac{2K}{x_K^3} G(x_K) - \frac{K}{x_K} f(x_K) - (1-K)\, f'(x_K)\right\} \varphi[g(K)x_K]$$

$$(4.7)\qquad \frac{\partial^2 R(K,x_K)}{\partial k^2} = \left\{1-C_o + \frac{K}{x_K} G(x_K) - F(x_K)\right\} \Big\{x_K\, g''(K)\, \varphi'[g(K)x_K]$$
$$+ x_K^2[g'(K)]^2\, \varphi''[g(K)x_K] + 2G(x_K)\, g'(K)\, \varphi'[g(K)x_K]\Big\}$$

$$(4.8) \qquad \frac{\partial^2 R(K,x_K)}{\partial x\ \partial k} = \left\{1-C_o + \frac{K}{x_K} G(x_K) - F(x_K)\right\}$$

$$\times\left\{g'(K)\ \varphi'[g(K)x_K] + x_K\ g(K)\ g'(K)\ \varphi''[g(K)x_K]\right\}$$

$$+ (1-K)\left\{\frac{1}{x_K} G(x_K) - f(x_K)\right\} g(K)\ \varphi'[g(K)x_K]$$

$$+ \left\{f(x_K) - \frac{1}{x_K^2} G(x_K)\right\} \varphi[g(K)x_K] \ .$$

Most of the necessary quantities will have been obtained in the solution of equations (4.5). Once the local maximum is confirmed, (4.4) can be verified by:

$$(4.9) \qquad \frac{G(x_k)}{k\ G(x_k) + (1-k)x_k^2\ f(x_k)} \begin{cases} \leq -\dfrac{g'(k)}{g(k)}\ , & K < k \leq 1 \\[2ex] \geq -\dfrac{g'(k)}{g(k)}\ , & 0 \leq k < K\ . \end{cases}$$

Even if (4.9) does not hold for all k, it may hold for k in some interval containing K, in which case no local maxima of $R(k,x)$ will occur for $k \neq K$ in the interval.

CHAPTER V

PROCEDURE AND EXAMPLES

1. Procedure for Solution.

To solve a particular guarantee problem of the type being considered, a procedure may be followed which takes up the various criteria which have been derived in the order of their difficulty of application. The following sequence ordered according to the various possible values of the parameters is an example: Whenever an optimality condition is satisfied, the search is terminated; and whenever a particular parameter value is shown not to be optimal, the remaining sub-steps for that value may be omitted.

(a) $x = 0$ (no guarantee given).

(1) If $\varphi(0) = 0$ or if $f(0) = 0$ and $\varphi'(0) > 0$, then $x = 0$ cannot be optimal (Cor. 1.1.1 of Ch. III).

(2) If $\varphi(0) > 0$, $f(0) > 0$, and $\varphi'(0) = 0$, then $R(\cdot,0)$ is a local maximum (Theorem 1.1 of Ch. III).

(3) If $\varphi(0) > 0$, $f(0) > 0$, and $\varphi'(0) > 0$, then the conditions of Theorem 1.1 of Ch. III must be checked.

(4) If $\varphi(0) > 0$, $f(0) = 0$, and $\varphi'(0) = 0$, then the conditions of Theorem 1.2 of Ch. III must be checked.

(5) If $R(\cdot,0)$ is a local maximum and $\frac{\partial R(k,x)}{\partial x} \leq 0$, for all (k,x), then $x = 0$ is optimal (Theorem 2.1 of Ch. III).

(b) $k = 1$.

(1) If $g(1) + g'(1) < 0$, $k = 1$ cannot be optimal (Cor. 3.1.2 of Ch. IV).

(2) If $g(1) + g'(1) > 0$, $R(1,x_1)$ is a local maximum (Theorem 3.1 of Ch. IV).

(3) If, when (2) holds, $\int_0^x y^2 f'(y)\,dy \leq 0$, $0 < x$, and $g(k) + (2-k)\,g'(k) > 0$, for $k < 1$, then $k = 1$ is optimal (Cor. 3.2.2 of Ch. IV). If these conditions hold for k such that $k_1 < k < 1$, but not for $0 < k < k_1$, then $R_1(1,x_1)$ is larger than $R(k,x)$ for k in the interval $(k_1,1)$.

(c) $k = 0$.

(1) If $\frac{x^2 f(x)}{G(x)} < -\frac{g(0)}{g'(0)}$, for all $x > 0$, then $k = 0$ cannot be optimal (Cor. 2.1.3 of Ch. IV).

(2) If $\frac{x^2 f(x)}{G(x)} > -\frac{g(0)}{g'(0)}$, for all $x > 0$, then $R(0,x_o)$ is a local maximum (Cor. 2.1.1 of Ch. IV).

(3) If (2) does not hold and, where x_o satisfies (1.4) of Ch. IV, $\frac{x_o^2 f(x_o)}{G(x_o)} < -\frac{g(0)}{g'(0)}$, then $k = 0$ cannot be optimal (Cor. 2.1.2 of Ch. IV).

(4) If (2) does not hold but $\frac{x_o^2 f(x_o)}{G(x_o)} > -\frac{g(0)}{g'(0)}$, then $R(0,x_o)$ is a local maximum (Theorem 2.1 of Ch. IV).

(5) If $R(0,x_o)$ is a local maximum and $k + (1-k)\frac{x^2 f(x)}{G(x)} \geq -\frac{g(k)}{g'(k)}$, for all (k,x), then $k = 0$ is optimal (Cor. 2.2.1 of Ch. IV). If these conditions hold for k such that $0 < k < k_o$, but not

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for $k_o < k < 1$, then $R(0,x_o)$ is larger than $R(k,x)$ for k in the interval $(0,k_o)$.

(d) If no optimality condition has been satisfied prior to this step, x_k must be computed for all k for which $R(k,x)$ has not been shown to be less than $R(\cdot,0)$, $R(1,x_1)$, or $R(0,x_o)$ by partial fulfillment of (a)(5), (b)(3), or (c)(5). These x_k must satisfy

$$\frac{\frac{k}{x_k^2} G(x_k) + (1-k) f(x_k)}{1-C_o + \frac{k}{x_k} G(x_k) - F(x_k)} = \frac{g(k)\,\varphi'[g(k)x_k]}{\varphi[g(k)x_k]} .$$

(1) If $R(1,x_1)$ is a local maximum and $k+(1-k)\frac{x_k^2 f(x_k)}{G(x_k)} \le -\frac{g(k)}{g'(k)}$, for $k < 1$, then $k = 1$ is optimal. If this condition holds for k such that $k_1 < k < 1$, then $R(1,x_1)$ is larger than $R(k,x)$ for k in the interval $(k_1,1)$.

(2) If $R(0,x_o)$ is a local maximum and $k+(1-k)\frac{x_k^2 f(x_k)}{G(x_k)} \ge -\frac{g(k)}{g'(k)}$, for $0 < k$, $k = 0$ is optimal. If this condition holds for k such that $0 < k < k_o$, then $R(0,x_o)$ is larger than $R(k,x)$ for k in the interval $(0,k_o)$.

(e) $0 < K < 1$.

(1) Solve (4.5) of Ch. IV for (K,x_K) and check (4.6), (4.7), and (4.8) of Ch. IV.

(2) If $R(K,x_K)$ is a local maximum and (4.9) of Ch. IV is satisfied, K is optimal.

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(f) If there is more than one local maximum, the value of each must be calculated and the greatest selected. This can occur by x_k being multiple valued or by local maxima for different values of k.

2. Solution of the Case $F(\xi) = 1-e^{-\xi}$ and $D(k,x) = \varphi_o + ab(k)\, x^j$.

A commonly encountered failure distribution is the negative exponential, $F(\xi) = 1-e^{-\xi}$. This distribution will be used as an example. For this distribution,

$$f(y) = e^{-y},\; f'(y) = -e^{-y} \text{ and } G(x) = 1-(1+x)e^{-x} . \tag{2.1}$$

The demand function will be specialized to the following form:

$$\varphi[g(k)x] = \varphi_o + ab(k)\, x^j , \qquad \varphi_o \geq 0,\; a > 0,\; j > 0 .$$

Then

$$\varphi(y) = \varphi_o + a\, y^j , \qquad \varphi'(y) = aj\, y^{j-1} ,$$

$$g(k) = [b(k)]^{\frac{1}{j}} \text{ and } g'(k) = \frac{1}{j}\, b'(k)\, [b(k)]^{\frac{1-j}{j}} . \tag{2.2}$$

Applying the procedure given in section 1, the following results are obtained:

(a) $x = 0$.

(1) If $\varphi_o = 0$, $x = 0$ cannot be optimal.

(2) $f(0) = 1$, therefore if $\varphi_o > 0$, $R(\cdot,0)$ may be a local maximum, depending on $\varphi'(0^+)$.

$$\varphi'(0^+) = \lim_{x \to 0^+} aj[b(k)]^{\frac{j-1}{j}} x^{j-1}$$

$$= \begin{cases} \infty \ , \ j < 1 \\ a \ , \ j = 1 \\ 0 \ , \ j > 1 \ ; \end{cases} \quad \text{therefore,}$$

if $j < 1$, $x = 0$ cannot be optimal. If $\varphi_o > 0$ and $j > 1$, $R(\cdot,0)$ is a local maximum; and, if $j = 1$, $R(\cdot,0)$ is a local maximum if $\varphi_o > 2a(1-C_o) \max\limits_{0 \leq k \leq 1} \frac{b(k)}{2-k}$.

(3) If $j = 1$,

$$\frac{\partial R(k,x)}{\partial x} = \{e^{-x}-C_o + \frac{k}{x}[1-(1+x)e^{-x}]\}\, ab(k)$$

$$- \{\frac{k}{x^2}[1-(1+x)e^{-x}] + (1-k)e^{-x}\}\{\varphi_o + ab(k)x\}$$

$$= \{e^{-x}-C_o-(1-k)xe^{-x}\}ab(k)-\{\frac{k}{x^2}[1-(1+x)e^{-x}]+(1-k)e^{-x}\}\, \varphi_o .$$

Hence $\frac{\partial R(k,x)}{\partial x} \leq 0$ if

$$\varphi_o \geq \frac{[1-(1-k)x]e^{-x}-C_o}{\frac{k}{x^2}[1-(1+x)e^{-x}]+(1-k)e^{-x}}\, ab(k) = \frac{1-(1-k)x-C_oe^{x}}{\frac{k}{x^2}[e^{x}-(1+x)]+(1-k)}\, ab(k).$$

The numerator decreases with x, while the denominator increases more rapidly than $k(\frac{1}{2} + \frac{x}{3!})$. Therefore the condition $\frac{\partial R(\cdot,0)}{\partial x} < 0$ is sufficient to insure that $\frac{\partial R(k,x)}{\partial x} < 0$, for all (k,x). Then if $j = 1$ and $R(\cdot,0)$ is a local maximum, $x = 0$ is optimal. This will not necessarily be true for $j > 1$.

(b) $k = 1$.

(1) $$g(1) + g'(1) = [b(1)]^{\frac{1}{j}} + \frac{1}{j} b'(1) [b(1)]^{\frac{1}{j} - 1}$$

$$= \frac{1}{j} [b(1)]^{\frac{1}{j} - 1} [jb(1) + b'(1)] .$$

Then, if $jb(1) < -b'(1)$, $k = 1$ cannot be optimal.

(2) If $jb(1) > -b'(1)$, $R(1,x_1)$ is a local maximum.

(3) $\int_0^x y^2 f'(y)\, dy = - \int_0^x y^2 e^{-y}\, dy < 0$ for $x > 0$, therefore if

$$jb(k) > (2-k)\, b'(k) , \qquad k < 1 ,$$

then $k = 1$ is optimal.

If this inequality holds for k in the interval $(k_1,1)$, no local maxima of $R(k,x)$ can occur there, even though $k = 1$ may not be optimal.

(c) $k = 0$.

(1) $$\frac{x^2 f(x)}{G(x)} = \frac{x^2 e^{-x}}{1-(1+x)e^{-x}} = \frac{1}{\frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots}$$; therefore, if

$2 < - \frac{jb(0)}{b'(0)}$, $k = 0$ cannot be optimal.

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(2) From (1.4) of Ch. II, $x_o^o = -\log C_o$, so that

$$\frac{x^2 f(x)}{G(x)} > \frac{(\log C_o)^2 C_o}{1-(1-\log C_o)C_o} . \quad \text{Then, if} \quad \frac{(\log C_o)^2 C_o}{1-(1-\log C_o)C_o} > -\frac{jb(0)}{b'(0)} ,$$

$R(0,x_o)$ is a local maximum.

(3) From (1.4) of Ch. IV, x_o must satisfy

$$\frac{e^{-x_o}}{e^{-x_o} - C_o} = \frac{ajb(k)\, x_o^{j-1}}{\varphi_o + ab(k)\, x_o^j} \quad \text{or} \quad (1-C_o\, e^{x_o}) j x_o^{j-1} - x_o^j = \frac{\varphi_o}{ab(k)} .$$

Then, if $\dfrac{x_o^2}{[e^{x_o}-(1+x_o)]} < -\dfrac{jb(0)}{b'(0)}$, $k = 0$ cannot be optimal.

(4) If $\dfrac{x_o^2}{[e^{x_o}-(1+x_o)]} > -\dfrac{jb(0)}{b'(0)}$, $R(0,x_o)$ is a local maximum.

(5) $$k + (1-k)\frac{x^2\, e^{-x}}{1-(1+x)e^{-x}} = k + (1-k)\frac{1}{\frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots}$$

$$\geq k + 2(1-k)\, e^{-x}$$

$$\geq k + 2(1-k)\, e^{-x_k^o}$$

$$\geq k + 2(1-k)\, e^{-x_1^o} ,$$

where $\dfrac{1}{x_1^o}(1-e^{-x_1^o}) = C_o$, because $x_1^o \geq x_k^o$, for $k < 1$.

Then, if $R(0,x_o)$ is a local maximum and

$k + 2(1-k)\, e^{-x_1^o} > - \frac{jb(k)}{b'(k)}$, for $0 < k$, $k = 0$ is optimal. If this inequality holds for k in the interval $(0,k_o)$, no local maxima of $R(k,x)$ can occur there, even though $k = 0$ may not be optimal.

(d) For the range of k, for which $R(k,x)$ has not been shown to be less than $R(\cdot,0)$, $R(1,x_1)$, or $R(0,x_o)$, x_k must satisfy

$$\frac{\frac{k}{x_k^2}\,[1-(1+x_k)e^{-x_k}] + (1-k)e^{-x_k}}{e^{-x_k} - C_o + \frac{k}{x_k}\,[1-(1+x_k)e^{-x_k}]} = \frac{ab(k)\; j\; x_k^{j-1}}{\varphi_o + a\, b(k)\, x_k^j} \;.$$

In order to exhibit a specific example, suppose $\varphi_o = 0$ and $j = 1$. Then x_k must satisfy

$$e^{-x_k}\,[1-(1-k)x_k] = C_o \;. \tag{2.3}$$

Newton's method is performed by iterating

$$x_k \approx x + \frac{[1-(1-k)x]-C_o e^x}{1+(1-k)(1-x)} \;.$$

Values of x_k for this example are shown in Table 1.

TABLE I

Values of x_k for $F(\xi) = 1-e^{-\xi}$ and $D(k,x) = ab(k)\,x$

$k \backslash C_o$	0	.1	.2	.4	.6	.8	1.0
0	1.0000	.7815	.6260	.4020	.2384	.1084	0
.2	1.2500	.9324	.7308	.4589	.2688	.1212	0
.4	1.6667	1.1436	.8706	.5319	.3071	.1372	0
.6	2.5000	1.4423	1.0587	.6274	.3569	.1579	0
.8	5.0000	1.8428	1.3066	.7531	.4225	.1854	0
1.0	∞	2.3026	1.6094	.9163	.5108	.2231	0

(1) If $R(k,x_1)$ is a local maximum of $R(k,x)$ and, for $k < 1$,

$$k + (1-k)\frac{x_k^2}{e^{x_k} - (1+x_k)} \leq -\frac{b(k)}{b'(k)},$$

then $k = 1$ is optimal. If this inequality holds for k such that $k_1 < k < 1$, then $R(1,x_1)$ is larger than $R(k,x)$ for k in the interval $(k_1,1)$, and no local maximum can occur there.

If $k + (1-k)\dfrac{x_k^2 f(x_k)}{G(x_k)}$ is denoted $\rho(k)$, values of $\rho(k)$ for the example being considered are given in Table II.

TABLE II

Values of $\rho(k)$ for $F(\xi) = 1-e^{-\xi}$ and $D(k,x) = ab(k)\ x$

$k \backslash C_o$	0	.1	.2	.4	.6	.8	1.0
0	1.392	1.515	1.605	1.741	1.844	1.929	2.0
.2	1.208	1.344	1.435	1.565	1.660	1.736	1.8
.4	1.034	1.174	1.278	1.397	1.480	1.546	1.6
.6	.888	1.065	1.144	1.242	1.308	1.358	1.4
.8	.804	.996	1.046	1.106	1.146	1.176	1.2
1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.0

(2) If $R(0,x_o)$ is a local maximum and, for $0 < k \leq 1$, $\rho(k) \geq -\dfrac{b(k)}{b'(k)}$, $k = 0$ is optimal. If this inequality holds for k such that $0 < k < k_o$, then $R(0,x_o)$ is greater than $R(k,x)$ for k in the interval $(0,k_o)$, and no local maximum can occur there.

(e) (1) The solution of the simultaneous equations (4.5) of Ch. IV now becomes equivalent to finding K such that

$$\rho(K) = \frac{b(K)}{b'(K)} \ .$$

Suppose that in this example, $C_o = .4$ and $\rho(.4) = -\dfrac{b(.4)}{b'(.4)}$, so that $(K,x_K) = (.4,\ .5319)$ is a possible local maximum.

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$$\frac{\partial^2 R(K,x_K)}{\partial x^2} = -2\left\{\frac{K}{x_K^2}\,[1-(1+x_K)e^{-x_K}] - (1-K)e^{-x_K}\right\} ab(K)$$

$$+\left\{\frac{2K}{x_K^3}\,[1-(1+x_K)e^{-x_K}] - \frac{K}{x_K}\,e^{-x_K} + (1-K)e^{-x_K}\right\} ab(K)\; x_K$$

$$= \quad [(1-K)\,x_K - (2-K)]\; e^{-x_K}\; ab(K)$$

$$= -\;.7519\; ab(.4) < 0\;.$$

$$\frac{\partial^2 R(K,x_K)}{\partial k^2} = \left\{e^{-x_K} - C_o + \frac{K}{x_K}\,[1-(1+x_K)\,e^{-x_K}]\right\} ab''(K)\; x_K$$

$$+\,2\left\{\frac{1}{x_K}\,[1-(1+x_K)\,e^{-x_K}]\right\} ab'(K)\; x_K$$

$$= \quad [.2627b''(.4) + .376b'(.4)]\cdot(.5319)a\;;$$

therefore $\dfrac{\partial^2 R(K,x_K)}{\partial k^2} < 0 \quad \text{if} \quad b''(.4) < -1.43b'(.4)\;.$

$$\frac{\partial^2 R(K,x_K)}{\partial x\,\partial k} = \left\{e^{-x_K} - C_o + \frac{K}{x_k}\,[1-(1+x_K)\,e^{-x_K}]\right\} ab'(K)$$

$$+\left\{\frac{1}{x_K}\,[1-(1+x_K)\,e^{-x_K}]\right\} ab(K)$$

$$-\left\{\frac{K}{x_K^2}\,[1-(1+x_K)\,e^{-x_K}] + (1-K)\,e^{-x_K}\right\} ab'(K)\; x_K$$

$$+\left\{\frac{-1}{x_K^2}\,[1-(1+x_K)\,e^{-x_K}] + e^{-x_K}\right\} ab(K)\; x_K$$

$$= -\;.4362\; ab'(.4)\;.$$

Then $\dfrac{\partial^2 R(K,x_K)}{\partial x^2} \cdot \dfrac{\partial^2 R(K,x_K)}{\partial k^2} - \left[\dfrac{\partial^2 R(K,x_K)}{\partial x\, \partial k}\right]^2 = [.147b''(.4)+.02b'(.4)]a^2 b'(.4)$

which is positive if $b''(.4) < -.136b'(.4)$. $-.136b'(.4) < -1.43b'(.4)$. Therefore, if $b''(.4) < -.136b'(.4)$, $R(.4, .5319)$ is a local maximum of $R(k,x)$.

(2) For this example, x_k is unique for any particular k, therefore $k = .4$ will be optimal if

$$-\frac{g(k)}{g'(k)} \begin{cases} < \rho(k), & .4 < k \leq 1 \\ > \rho(k), & 0 \leq k < .4 \end{cases}$$

If these inequalities hold for k in an interval (k_0,k_1), where $k_0 < .4 < k_1$, then $R(.4, .5319)$ is larger than $R(k,x)$ for k in the interval (k_0,k_1) even though $k = .4$ may not be optimal.

3. <u>Solution of the Case</u> $F(\xi) = 1-e^{-\xi}$ <u>and</u> $D(k,x) = a(1-dk)\,x$.

As a more specific example, let $f(y) = e^{-y}$ and $D(k,x) = a(1-dk)x$. This is a further specialization of the example of section 2; therefore, the following results are immediate from the conditions obtained there.

(a) $x = 0$ cannot be optimal.

(b) $b(1) = 1-d$ and $b'(k) = -d$ for all k; therefore, $k = 1$ is optimal if $d \leq 1/2$ and cannot be optimal if $d > 1/2$.

(c) $b(0) = 1$, therefore if $\dfrac{1}{d} < \dfrac{C_o(\log C_o)^2}{1-C_o(1-\log C_o)} = \rho(0)$, $R(0, -\log C_o)$ is a local maximum of $R(k,x)$.

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(d) If $R(0, -\log C_o)$ is a local maximum of $R(k,x)$, then

$$-\frac{b(k)}{b'(k)} = \frac{1}{d} - k \leq \frac{1}{d} \leq \rho(0) \leq \rho(k);$$ therefore, $k = 0$ is optimal.

(e) (1) If $d = .5568$ and $C_o = .4$, $\frac{1}{d} - .4 = -\frac{b(.4)}{b'(.4)} = \rho(.4)$. And, $b''(k) = 0$ for all k, therefore $R(.4, .5319)$ is a local maximum of $R(k,x)$.

(2) $\frac{1}{d} - k \leq \frac{1}{d} - .4 = \rho(.4) \leq \rho(k)$ for $.4 \leq k$, and

$\frac{1}{d} - k \geq \frac{1}{d} - .4 = \rho(.4) \geq \rho(k)$ for $.4 \leq k$; therefore,

if $C_o = .4$ and $d = .5568$, $k = .4$ is optimal.

(f) Figure 3 shows graphs of the optimal values of k on the space of parameters C_o versus d. Optimal values of x can be obtained from Table 1.

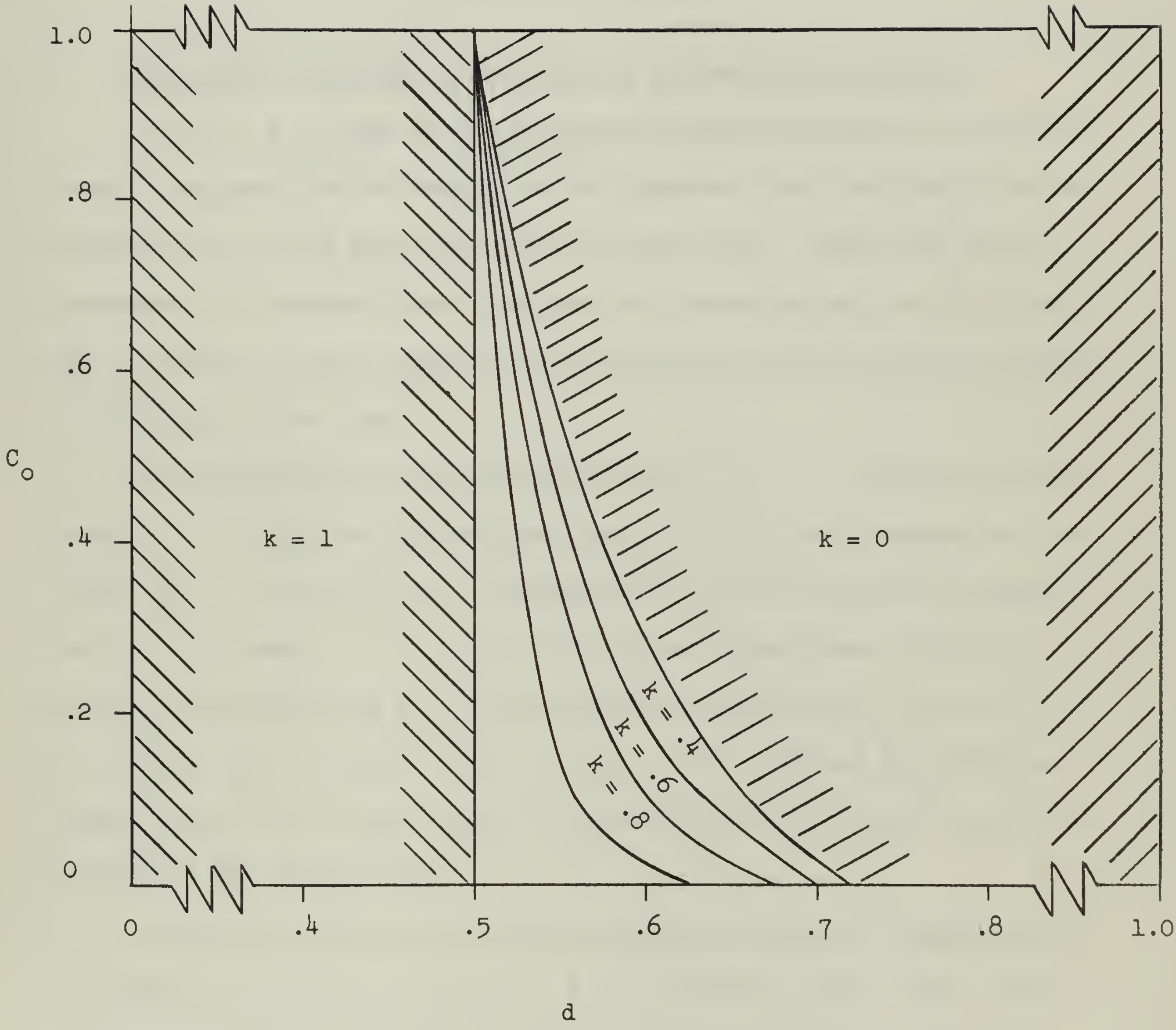

FIGURE 3

Optimal Values of k for $F(\xi) = 1-e^{-\xi}$

and $D(k,x) = a(1-dk)x$

CHAPTER VI

ALTERNATIVE MODEL

1. Alternative Guarantee Policy and Its Mathematical Structure.

In the $k = 0$ case of the alternative guarantee policy, the failed item is replaced free of charge but the guarantee runs only until the expiration time of the guarantee of the initial sale. Under this policy, purchases by a customer form a sequence of renewal cycles, with the number of items per cycle depending on the guarantee length and the lifetime distribution of the item.

The generalization, which includes both this $k = 0$ policy and the same $k = 1$ policy as the previous generalization, is to charge the customer $\frac{k\xi}{x} \cdot C$ for the first replacement item, and to extend the guarantee by $k\xi$, where $\xi < x$ is the life of the failed item. For this policy, the second item in a cycle would be guaranteed for a period $x-(1-k)\xi_1$, the third for $x-(1-k)(\xi_1+\xi_2)$, etc. Because the guarantee length varies from item to item, this model proves to be much less tractable than the previous one.

Consider a typical cycle from the sequence of cycles formed by the purchases of a single customer over a long period of time. Each cycle will commence with an initial sale and terminate with the first item which lasts beyond its associated guarantee. Then if the N-th item is the final item of the cycle,

$$\xi_{N-1} < x-(1-k)\ (\xi_1 + \cdots + \xi_{N-2}) \quad \text{and} \quad \xi_N \geq x-(1-k)\ (\xi_1 + \cdots + \xi_{N-1})\ .$$

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If the profit for the cycle is designated by Y,

$$Y = (C-C_o) + (\frac{k\xi_1}{x} C-C_o) + (\frac{k\xi_2}{x} C-C_o) + \cdots + (\frac{k\xi_{N-1}}{x} C-C_o)$$

$$= C[1 + \frac{k}{x}(\xi_1 + \cdots + \xi_{N-1})] - N\,C_o$$

$$= C(1 + \frac{k}{x} S_{N-1}) - N\,C_o \quad ,$$

where for any positive integer r, $S_r = \xi_1 + \xi_2 + \cdots + \xi_r$. The length of the cycle is thus S_N. As the process operates over a long period of time, the sequence of such cycles forms a renewal process. The i-th time interval of the renewal sequence will be of length S_{N_i}, and a profit of Y_i will be obtained during that interval. Then $\{S_{N_i}\}$ and $\{Y_i\}$, for $i = 1, 2, \ldots$, are both sequences of independent, identically distributed random variables, and Y_h is independent of S_{N_i} for $h \neq i$. For fixed $T > 0$, let $S_{N_0} = Y_o = 0$, and let $J \geq 0$ be such that $S_{N_0} + S_{N_1} + \cdots + S_{N_{J-1}} < T \leq S_{N_0} + \cdots + S_{N_J}$. Let $H_1(T)$ be the sum of the profits obtained during a period of time T beginning at the origin of the process. If it is assumed that the $\{Y_i\}$ are obtained at the beginnings of the associated time intervals, some profits may be included which do not belong in $H_1(T)$. If the sum thus formed is designated $H_1^*(T)$, then $H_1^*(T)$ is an upper bound for $H_1(T)$. Here

$$H_1^*(T) = \begin{cases} Y_1 , & S_{N_1} \geq T \\ Y_1 + Y_2 + \cdots + Y_J , & S_{N_1} < T . \end{cases}$$

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For $S_{N_1} < T$, the cycles of the sequence remaining after S_{N_1} form an analogous renewal process. Profits for the remaining period of length $T - S_{N_1}$ commence with Y_2 and can be designated $H_2^*(T - S_{N_1})$. Then

$$E\{H_1^*(T)\} = \int_0^\infty E\{H_1^*(T)|S_{N_1} = u\} \, dF_{S_N}(u)$$

$$= E\{Y_i\} + \int_0^T E\{H_2^*(T-u)\} \, dF_{S_N}(u) .$$

Since the $\{S_{N_i}\}$ and $\{Y_i\}$ are sequences of independent, identically distributed random variables, $E\{H_i^*(\circ)\}$ does not depend on the index i, and all the subscripts can be dropped. It follows that $E\{H^*(T)\}$ satisfies the generalized renewal equation

$$E\{H^*(T)\} = E\{Y\} + \int_0^T E\{H^*(T-u)\} \, dF_{S_N}(u)$$

with the solution

$$E\{H^*(T)\} = E\{Y\} \, [1 + M_{S_N}(T)] .$$

Proceeding as in Section 1 of Chapter II gives

$$R_1^*(k,x) = \frac{E\{Y\}}{E\{S_N\}} ,$$

an upper bound for $R_1(k,x)$, the expected long-term average rate of profit per single customer.

If it is assumed that the $\{Y_i\}$ are obtained at the ends of the associated time intervals, rather than at the beginnings, an analogous quantity $R_1^{**}(k,x)$ is obtained. Under this assumption, some profits which actually belong in $H_1(T)$ are omitted, so that $R_1^{**}(k,x)$ is a lower bound for $R_1(k,x)$. Following an argument similar to that used to evaluate $R_1^*(k,x)$, it is found that $R_1^{**}(k,x) = R_1^*(k,x)$; and therefore

$$R_1(k,x) = \frac{E\{Y\}}{E\{S_N\}}$$

$$= \frac{E\{C(1 + \frac{k}{x} S_{N-1}) - N C_o\}}{E\{S_N\}} .$$

By a result of Wald [6],

$$E\{S_N\} = E\{\xi_1 + \xi_2 + \cdots + \xi_N\} = E\{\xi\} \cdot E\{N\} .$$

Without loss of generality, it can be assumed that C_o is expressed as a fraction of C and that $C = E\{\xi\} = 1$, so that

$$R_1(k,x) = \frac{1}{E\{N\}} + \frac{k}{x} \frac{E\{S_{N-1}\}}{E\{N\}} - C_o . \tag{1.1}$$

The quantity $E\{N\}$ is a function of k and x and will be denoted $u(k,x)$. Then

$$E\{N|\xi_1 = y\} = \begin{cases} 1 & , \quad y \geq x \\ 1+u(k, x-(1-k)y) & , \quad y < x ; \end{cases}$$

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from which it follows that

$$(1.2) \qquad u(k,x) = 1 + \int_0^x u(k,\, x-(1-k)y)\, dF_\xi(y) \ .$$

For $k = 0$, this is a generalized renewal equation, and

$$u(0,x) = 1 + M_\xi(x) \ .$$

For $0 < k < 1$, (1.2) is more difficult to solve than a renewal equation, and solutions in closed form have not been obtained for failure distributions of interest. Furthermore, asymptotic solutions for large x are not relevant to the guarantee problem under consideration.

2. <u>Expected Number of Items per Cycle for the Case</u> $F(\xi) = 1-e^{-\xi}$.

For the type of guarantee policy discussed in the foregoing section, consider the special case where $F(\xi) = 1-e^{-\xi}$. By substituting $e^{-y}\,dy$ for $dF_\xi(y)$ in (1.2) and differentiating with respect to x, the following equivalent differential equation is obtained.

$$(2.1) \qquad (1-k)\,\frac{\partial u(k,x)}{\partial x} + ke^{-x}\,u(k,kx) = 1 \ .$$

If $U(k,s)$ is the Laplace transform of $u(k,x)$ with respect to x, where k is held fixed,

$$(2.2) \qquad U(k,s) = \frac{1}{(1-k)s^2} + \frac{1}{s} - \frac{1}{(1-k)s}\,U\!\left(\frac{s+1}{k}\right) \ .$$

By assuming an initial trial solution $U_1(k,s) = U(k, \frac{s+1}{k})$, a process of iteration gives

$$(2.3) \qquad U_n(k,s) = \sum_{j=0}^{n} \left\{ \left(\frac{-k}{1-k}\right)^j \frac{1}{s^j(s+\gamma_j)} + \frac{(-k^2)^j}{(1-k)^{j+1}} \frac{1}{s^j(s+\gamma_j)^2} \right\},$$

where $\gamma_0 = 0$ and $\gamma_j = \sum_{i=1}^{j} k^{i-1}$, for $j = 1, 2, \ldots$. That $\lim_{n \to \infty} U_n(k,s)$ satisfies (2.2) can be verified by substitution. Unfortunately, this limit is not recognized as the Laplace transform of any function of closed form. Term by term inversion gives the following series representation in which higher order terms can be neglected only if k is near 0:

$$u(k,x) = 1 + \frac{x}{1-k} + \left(\frac{k}{1-k}\right)^2 \left(x + \frac{1}{k}\right) e^{-x}$$

$$+ \left(\frac{k}{1-k}\right)^3 \left(\frac{k}{1+k} x + \frac{1+k+k^2}{k(1+k)^2}\right) e^{-(1+k)x} + \cdots .$$

Since it is necessary to use $u(k,x)$ in equations which will be solved for k and x, this series solution is of little use.

If the distribution of times of failure is the gamma distribution with density function $f(y) = 4y\, e^{-2y}$, the Laplace transform of $u(k,x)$ will include four infinite sums of terms; therefore, it is conjectured that equally complicated results will derive from other forms of $F(\xi)$ of interest.

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3. <u>Comparison of the</u> $k = 0$ <u>and</u> $k = 1$ <u>Cases</u>.

Even though results for general k have not been obtained for the alternative generalized guarantee policy, a direct comparison of the $k = 0$ and $k = 1$ cases can be made. From (1.1),

$$R_1(0,x) = \frac{1}{1 + M(x)} - C_o , \tag{3.1}$$

and if $D(0,x) = \varphi[g(0)x]$, as in the previous model,

$$R(0,x) = \left(\frac{1}{1 + M(x)} - C_o\right) \varphi[g(0)x] . \tag{3.2}$$

(a) For $R(0,x)$ to have a maximum at $x = 0$, a necessary condition is $\frac{\partial R(0,0^+)}{\partial x} \leq 0$, for $0 \leq k \leq 1$, with strict inequality being sufficient. Assuming that $F(x)$ is continuous,

$$\frac{\partial R(0,0^+)}{\partial x} = \left\{\frac{1}{1 + M(0^+)} - C_o\right\} g(0)\ \varphi'(0^+) - \frac{M'(0^+)}{[1 + M(0^+)]^2} .$$

$M(0) = 0$ and $M'(x) = f(x) + \int_0^x M'(x-\xi)\ f(\xi)\ d\xi$; therefore

$$\frac{\partial R(0,0^+)}{\partial x} = (1-C_o)\ g(0)\ \varphi'(0) - f(0)\ \varphi(0) ,$$

and $R(0,0)$ will be a maximum of $R(0,x)$ if

$$\frac{f(0)}{1-C_o} > \frac{g(0)\ \varphi'(0)}{\varphi(0)} .$$

(b) For $R(0,x)$ to have a local maximum at $x > 0$, it is necessary that $\frac{\partial R(0,x)}{\partial x} = 0$ and $\frac{\partial^2 R(0,x)}{\partial x^2} \leq 0$, with strict inequality being sufficient.

$$\frac{\partial R(0,x)}{\partial x} = \left\{\frac{1}{1+M(x)} - C_o\right\} g(0)\, \varphi'[g(0)x] - \frac{M'(x)}{[1+M(x)]^2}\, \varphi[g(0)x] ,$$

therefore

$$(3.3) \qquad \frac{M'(x)}{[1+M(x)][1-C_o(1+M(x))]} = \frac{g(0)\, \varphi'[g(0)x]}{\varphi[g(0)x]}$$

is necessary.

To compare this policy with the $k = 1$ policy, (3.3) must be solved for x_o. That $R(0,x_o)$ is a maximum of $R(0,x)$ is confirmed by showing that $\frac{\partial^2 R(0,x_o)}{\partial x^2} < 0$. Then $R(0,x_o)$ is calculated and compared directly with the $R(1,x_1)$ obtained from the first $k = 1$ guarantee model.

(c) Again taking the example $F(\xi) = 1-e^{-\xi}$ and $D(k,x) = a\, b(k)\, x$, it is seen that $M(x) = x$ and $M'(x) = 1$ so that (3.3) becomes

$$\frac{1}{(1+x_o)[1-C_o(1+x_o)]} = \frac{a\, b(0)}{a\, b(0) x_o} ,$$

and $x_o = \sqrt{\frac{1}{C_o}} - 1$. Furthermore,

$$\frac{\partial^2 R(0,x_o)}{\partial x^2} = \left\{\frac{1}{1 + M(x_o)} - C_o\right\} (g(0))^2 \varphi''[g(0)x_o]$$

$$- 2 \frac{M'(x_o)}{[1 + M(x_o)]^2} g(0) \varphi'[g(0)x_o]$$

$$+ \left\{\frac{2[M'(x_o)]^2}{[1-M(x_o)]^3} - \frac{M''(x_o)}{[1 + M(x_o)]^2}\right\} \varphi[g(0)x_o]$$

$$= - \frac{2}{(1 + x_o)^2} \cdot a\ b(0) + \frac{2}{(1 + x_o)^3} a\ b(0)\ x_o$$

$$= \frac{2a\ b(0)}{(1 + x_o)^2} \left(\frac{x_o}{1 + x_o} - 1\right) < 0 \ .$$

Therefore,

$$R(0, \sqrt{\frac{1}{C_o}} - 1) = ab(0)(1 - \sqrt{C_o}\ )^2$$

is a maximum of $R(0,x)$. From section 2.d. of Ch. V, it is found that $R(1,x_1)$ is a maximum of $R(1,x)$ if

$$R(1,x_1) = a\ b(1)[1-C_o(1 - \log C_o)] \ .$$

The value $k = 1$ is preferred to $k = 0$ if $b(1)[1-C_o(1 - \log C_o)] >$ $> b(0)(1 - \sqrt{C_o}\ )^2$ or

$$\frac{b(1)}{b(0)} > \frac{(1 - \sqrt{C_o}\ )^2}{1 - C_o(1 - \log C_o)} = \theta(C_o) \ .$$

Values of $\theta(C_o)$ are shown in Table III.

TABLE III

Values of $\theta(C_o)$ for $F(\xi) = 1-e^{-\xi}$, $D(k,x) = a\ b(k)x$

C_o	$\theta(C_o)$
0	1.000
.2	.626
.4	.579
.6	.543
.8	.519
1.0	.500

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